PROBLEMAS DE ELECTROMAGNETISMO

Curso 2020-21

1.- Determinar el gradiente de la función escalar

$$\varphi(x,y,z) = 17x - \frac{2xy}{z} + y^2 + z^2$$

así como su valor numérico en el punto (1,2,3).

$$\nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$$

$$\nabla\phi(x,y,z) = \left(17 - \frac{2y}{z}\right)\hat{i} + \left(-\frac{2x}{z} + 2y\right)\hat{j} + \left(+\frac{2xy}{z^2} + 2z\right)\hat{k}$$

$$\nabla\phi\Big|_{(1,2,3)} = \left(17 - \frac{4}{3}\right)\hat{i} + \left(-\frac{2}{3} + 4\right)\hat{j} + \left(\frac{4}{9} + 6\right)\hat{k} =$$

$$= \frac{47}{3}\hat{i} + \frac{10}{3}\hat{j} + \frac{58}{9}k = \nabla\phi\Big|_{(1,2,3)}$$

2.- Hallar el valor de $\vec{\nabla}\left(\frac{1}{r}\right)$ siendo $\vec{r}$ el vector de posición de un punto genérico del espacio de coordenadas (x, y, z).

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$r = |\vec{r}| = \sqrt{x^2+y^2+z^2}$$

$$\vec{\nabla}\left(\frac{1}{r}\right) = \frac{\partial}{\partial x}\left(\frac{1}{r}\right)\hat{i} + \frac{\partial}{\partial y}\left(\frac{1}{r}\right)\hat{j} + \frac{\partial}{\partial z}\left(\frac{1}{r}\right)\hat{k}$$

$$\frac{\partial}{\partial x}\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right) = \frac{-\cancel{2}x / \cancel{2}\sqrt{x^2+y^2+z^2}}{x^2+y^2+z^2} = \frac{-x}{(x^2+y^2+z^2)^{3/2}}$$

$$y = \sqrt[n]{u} = u^{1/n} \Rightarrow y' = \frac{1}{n} u' \cdot u^{\frac{1-n}{n}} ;$$

$$\frac{1}{n} - 1 = \frac{1-n}{n}$$

$$\text{Si } n = 2 \Rightarrow y' = \frac{1}{2} u' \cdot u^{-1/2} = \frac{u'}{2\sqrt{u}}$$

$$\vec{\nabla}\left(\frac{1}{r}\right) = \frac{-(x\cdot\hat{\imath} + y\cdot\hat{\jmath} + z\cdot k)}{(x^2+y^2+z^2)^{3/2}} = \frac{-\vec{r}}{|r|^3} = \frac{-\hat{r}}{|r|^2}$$

Si lo hubiéramos hecho en esféricas.-

$$\vec{\nabla}\phi = \frac{\partial\phi}{\partial r}\vec{r} + \frac{1}{r}\frac{\partial\phi}{\partial\theta}\vec{\theta} + \frac{1}{r\,\mathrm{sen}\,\theta}\frac{\partial\phi}{\partial\varphi}\vec{\varphi}$$

$$\vec{\nabla}\left(\frac{1}{r}\right) = \frac{-1}{|r|^2}\hat{r} + 0\,\hat{\theta} + 0\,\hat{\varphi}$$

3.- Calcular el valor de $\vec{\nabla} \wedge \left(\frac{\vec{r}}{r^3}\right)$ en coordenadas cartesianas.

$$\vec\nabla \wedge \vec H = \begin{vmatrix} \hat i & \hat j & \hat k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ H_x & H_y & H_z \end{vmatrix}$$

Sea $\vec H = \dfrac{\vec r}{r^3} = \underbrace{\dfrac{x}{(x^2+y^2+z^2)^{3/2}}}_{H_x}\,\hat i + \underbrace{\dfrac{y}{(x^2+y^2+z^2)^{3/2}}}_{H_y}\,\hat j + \underbrace{\dfrac{z}{(x^2+y^2+z^2)^{3/2}}}_{H_z}\,\hat k$

$$\vec\nabla \wedge \vec H = \left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right)\hat i + \left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right)\hat j + \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right)\hat k$$

$$\frac{\partial H_z}{\partial y} \equiv \frac{\partial}{\partial y}\left(\frac{z}{(x^2+y^2+z^2)^{3/2}}\right) = \frac{-z \cdot \frac{3}{\cancel{2}} \cdot \cancel{2}y \cdot (x^2+y^2+z^2)^{1/2}}{(x^2+y^2+z^2)^3}$$

$$= \frac{-3zy}{(x^2+y^2+z^2)^{5/2}}$$

Las demas por simetría

$$\vec{\nabla}\wedge\vec{H} = \frac{-3zy-(-3zy)}{(x^2+y^2+z^2)^{5/2}}\hat{i} + \frac{-3xz+3xz}{(x^2+y^2+z^2)^{5/2}}\hat{j} + \frac{-3xy+3xy}{(x^2+y^2+z^2)^{1/2}}\hat{k}$$

$$= \vec{0}$$

NOTA.- En coord esféricas

$$\text{rot}\,\vec{H} = \cancel{\frac{1}{r\,\text{sen}\theta}}\ \frac{1}{r}\left[\frac{1}{\text{sen}\theta}\frac{\partial H_r}{\partial \varphi}\right.$$

$$= 0 \quad \text{directamente} \quad H = (H_r, 0, 0) = \left(\frac{\vec{r}}{|r|^3}, 0, 0\right)$$

4.- Calcular el operador rotacional del campo vectorial

$$\vec{F}(x,y,z) = 2xy\,\hat{x} + y^2 z\,\hat{y} + x^2 y^2\,\hat{z}$$

$$\vec{\nabla}\times\vec{H} = \left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right)\hat{i} + \left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right)\hat{j} + \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right)\hat{k}$$

$$F_x = 2xy$$

$$F_y = y^2 \cdot z$$

$$F_z = x^2 y^2$$

$$\vec{\nabla}\times\vec{F} = \left(2x^2y - y^2\right)\hat{i} + \left(0 - 2xy^2\right)\hat{j} + \left(0 - 2x\right)\hat{k} =$$

$$= \left(2x^2y - y^2\right)\hat{i} - 2xy^2\,\hat{j} - 2x\,\hat{k} = \underline{\left(2x^2y - y^2, -2xy^2, -2x\right) = \vec{\nabla}\wedge\vec{F}}$$

5.- Demostrar el cumplimiento de las identidades vectoriales

$$\vec{\nabla}\cdot\left(\varphi\vec{A}\right)=\vec{A}\cdot\vec{\nabla}\varphi+\varphi\,\vec{\nabla}\cdot\vec{A}$$

$$\vec{\nabla}\wedge\left(\varphi\vec{A}\right)=\varphi\,\vec{\nabla}\wedge\vec{A}-\vec{A}\wedge\vec{\nabla}\varphi$$

siendo φ una función escalar, y, $\vec{A}$ $(x,\ y,\ z)$ una de tipo vectorial de las coordenadas cartesianas.

$$\varphi\cdot\vec{A}=\varphi\left(A_x\,\hat{i}+A_y\,\hat{j}+A_z\,\hat{k}\right)=\left(\varphi A_x,\ \varphi A_y,\ \varphi A_z\right)$$

$$\underline{\underline{\vec{\nabla}\cdot(\varphi\cdot\vec{A})}}=\frac{\partial(\varphi A_x)}{\partial x}+\frac{\partial(\varphi A_y)}{\partial y}+\frac{\partial(\varphi A_z)}{\partial z}=\varphi\cdot\frac{\partial A_x}{\partial x}+A_x\frac{\partial\varphi}{\partial x}+\ldots$$

$$\underline{\underline{\vec{A}\cdot\vec{\nabla}\varphi}}=\left(A_x,A_y,A_z\right)\begin{pmatrix}\partial\varphi/\partial x\\ \partial\varphi/\partial y\\ \partial\varphi/\partial z\end{pmatrix}=A_x\frac{\partial\varphi}{\partial x}+A_y\frac{\partial\varphi}{\partial y}+A_z\cdot\frac{\partial\varphi}{\partial z}$$

$$\underline{\underline{\varphi\cdot\vec{\nabla}\cdot\vec{A}}}=\varphi\left(\frac{\partial A_x}{\partial x}+\frac{\partial A_y}{\partial y}+\frac{\partial A_z}{\partial z}\right)=\varphi\frac{\partial A_x}{\partial x}+\varphi\frac{\partial A_y}{\partial y}+\varphi\frac{\partial A_z}{\partial z}\quad\#$$

$$\vec{\nabla} \wedge (\varphi \cdot \vec{A}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \varphi A_x & \varphi A_y & \varphi A_z \end{vmatrix} = \hat{i}\left(A_z \frac{\partial \varphi}{\partial y} + \varphi \frac{\partial A_z}{\partial y} - A_y \frac{\partial \varphi}{\partial z} - \varphi \frac{\partial A_y}{\partial z}\right)$$

$$+ \hat{j}\left(A_x \frac{\partial \varphi}{\partial z} + \varphi \frac{\partial A_x}{\partial z} - A_z \frac{\partial \varphi}{\partial x} - \varphi \frac{\partial A_z}{\partial x}\right) + \hat{k}\left(A_y \frac{\partial \varphi}{\partial x} + \varphi \frac{\partial A_y}{\partial x} - A_x \frac{\partial \varphi}{\partial y} - \varphi \frac{\partial A_x}{\partial y}\right)$$

$$\varphi \vec{\nabla} \wedge \vec{A} = \varphi\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{i} + \varphi\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\hat{j} + \varphi\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{k}$$

$$\vec{A} \wedge \nabla \varphi = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ \frac{\partial \varphi}{\partial x} & \frac{\partial \varphi}{\partial y} & \frac{\partial \varphi}{\partial z} \end{vmatrix} = \hat{i}\left(A_y \frac{\partial \varphi}{\partial z} - A_z \frac{\partial \varphi}{\partial y}\right) + \hat{j}\left(A_z \frac{\partial \varphi}{\partial x} - A_x \frac{\partial \varphi}{\partial z}\right)$$

$$+ \hat{k}\left(A_x \frac{\partial \varphi}{\partial y} - A_y \frac{\partial \varphi}{\partial x}\right)$$

FT

6.- Comprobar que $\nabla^2\left(\frac{1}{r}\right) = 0$ si $r \neq 0$. Evaluar $\int_V \nabla^2\left(\frac{1}{r}\right) dV$ para un volumen cerrado que incluya el punto $r = 0$.

$$\nabla^2\left(\frac{1}{r}\right) = \vec{\nabla}\left(-\frac{\vec{r}}{r^3}\right) = -\vec{\nabla}\left(\frac{x}{(x^2+y^2+z^2)^{3/2}}, \frac{y}{(x^2+y^2+z^2)^{3/2}}, \frac{z}{(x^2+y^2+z^2)^{3/2}}\right)$$

$$\frac{\partial}{\partial x}\left(\frac{x}{(x^2+y^2+z^2)^{3/2}}\right) = \frac{(x^2+y^2+z^2)^{3/2}\cdot 1 - x\cdot\frac{3}{2}(x^2+y^2+z^2)^{1/2}\cdot 2x}{(x^2+y^2+z^2)^3} =$$

$$= \frac{(x^2+y^2+z^2)^{3/2} - 3x^2(x^2+y^2+z^2)^{1/2}}{(x^2+y^2+z^2)^3}$$

$$\frac{\partial}{\partial y}\left(\frac{y}{(\quad)^{3/2}}\right) = \frac{(x^2+y^2+z^2)^{3/2} - 3y^2(x^2+y^2+z^2)^{1/2}}{(\quad)^3}$$

$$\frac{\partial}{\partial z}\left(\frac{z}{(\ )^{3/2}}\right) = \frac{(\ )^{3/2} - 3z^2(\ \ \ )^{1/2}}{(\ )}$$

Sumando:

$$\vec{\nabla}\left(\frac{\vec{r}}{r^3}\right) = \frac{3(x^2+y^2+z^2)^{3/2} - 3(x^2+y^2+z^2)^{3/2}}{(x^2+y^2+z^2)^{3/2}} = 0 \qquad \text{si } r \neq 0$$

"No está definido en el origen." callar eso

$$\int_v \nabla^2\left(\frac{1}{r}\right) dv = \int_v \vec{\nabla}\cdot\left[\vec{\nabla}\left(\frac{1}{r}\right)\right] dv \overset{\text{Th. Diverg.}}{=} \int_s \vec{\nabla}\left(\frac{1}{r}\right)\cdot d\vec{s} =$$

$$= -\int_s \frac{\vec{r}\cdot d\vec{s}}{r^3} = -\int_{s_c} d\Omega = -4\pi$$

(si r está dentro de s_c y 0 si está fuera)

Se puede escribir

$$\nabla^2\left(\frac{1}{r}\right) = -4\pi\,\delta(r) \rightarrow \boxed{\delta(r) = -\frac{1}{4\pi}\nabla^2\left(\frac{1}{r}\right)}$$

6.- Comprobar que $\nabla^2\left(\frac{1}{r}\right) = 0$ si $r \neq 0$. Evaluar $\int_V \nabla^2\left(\frac{1}{r}\right) dV$ para un volumen cerrado que incluya el punto $r = 0$.

En esféricas

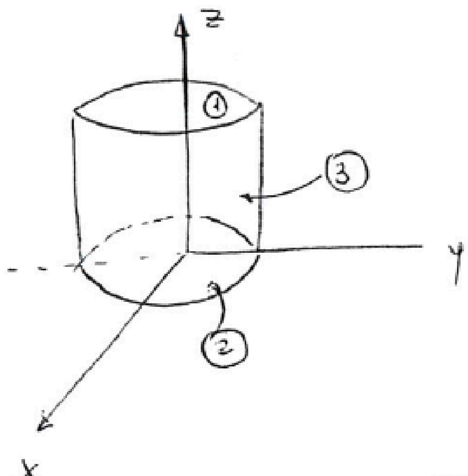
$$\nabla^2\left(\frac{1}{r}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}(1/r)\right) + 0 + 0 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(\cancel{r^2}\left(-\frac{1}{\cancel{r^2}}\right)\right) = 0$$

si $r \neq 0$

7.- Calcular el flujo del campo vectorial

$$\vec{A}(x,y,z) = z\,\hat{\mathrm{x}} + x\,\hat{\mathrm{y}} + 3y^2z\,\hat{\mathrm{z}}$$

a través de la superficie del cilindro definido por $x^2 + y^2 = 4$ y los planos $z = 0$, y, $z = 2$.



$$\oint \vec{A}\cdot\vec{dS} = \underbrace{\int \vec{A}\cdot\vec{dS_1}}_{1} + \underbrace{\int \vec{A}\cdot\vec{dS_2}}_{2} + \int \vec{A}$$

$$\vec{dS_1} = \hat{k}\, dS_1$$

$$1 = \int_{S_1} \left(z\,\hat{i} + x\,\hat{j} + 3y^2z\,\hat{k}\right)\cdot\hat{k}\, dS_1 =$$

$$= \int_{S_1} 3y^2z\, dS_1 = \int 3y^2z\, dx\, dy =$$

$y^2 = 4 - x^2$

$y = \sqrt{4-x^2}$

$$= \int_{x=-2}^{x=+2} \int_{-\sqrt{4-x^2}}^{+\sqrt{4-x^2}} 3y^2 z \, dy \, dx = \int_{x=-2}^{x=+2} \cancel{3} \cdot \frac{1}{\cancel{3}} y^3 z \Big|_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dx =$$

$$= \overset{z}{\underset{2}{\shortparallel}} \int \left(\left(\sqrt{4-x^2}\right)^3 - \left(-\sqrt{4-x^2}\right)^3 \right) dx = 4 \underbrace{\int_{-2}^{+2} \left(\sqrt{4-x^2}\right)^3 dx}_{} = 24\pi$$

NOTA.-

$$\int (a^2-x^2)^{3/2} dx = \frac{x(a^2-x^2)^{3/2}}{4} + \frac{3a^2x\sqrt{a^2-x^2}}{8} + \frac{3}{8}a^4 \operatorname{arcsen}\left(\frac{x}{a}\right) = \sqrt{}$$

$$\int_{-2}^{+2} (4-x^2)^{3/2} dx = \frac{3}{8} 2^4 \left[\operatorname{arc \cdot sen}(1) - \operatorname{arcsen}(-1)\right] = 6\left[\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right] = 6\pi$$

notar $a = 2$

② $d\vec{S}_2 = -\vec{k}\, dS_2 = -\hat{k}\, dx\, dy$

$$② = \int_{S_2} (z\hat{i} + x\hat{j} + 3y^2 z\,\hat{k})(-\vec{k}\, dx\, dy) = \int_x \int_y -3y^2 z\, dx\, dy = 0$$

$z = 0$

$\wedge \quad S_2 \Rightarrow z = 0$

③ Calculemos un vector $\perp$ a la superficie lateral.-

$$\vec{N} = \vec{\nabla}(x^2+y^2) = 2x\hat{i} + 2y\hat{j}$$

$$\hat{n} = \frac{2x\hat{i} + 2y\hat{j}}{\sqrt{4x^2+4y^2}} = \frac{x\hat{i} + y\hat{j}}{\sqrt{x^2+y^2}} = \frac{x\hat{i} + y\hat{j}}{2}$$

ya que $x^2+y^2=4$

$$③ = \int \underbrace{(z\hat{i} + x\hat{j} + 3y^2z\hat{k})\left(\frac{x}{2}\hat{i} + \frac{y}{2}\hat{j}\right)} \frac{4\,\overbrace{dS_y}^{dx\,dz}}{y} =$$

$$\frac{xz}{2} + \frac{xy}{2}$$

Si proyectamos ~~a la Superficie~~ segun el eje y

$$\hat{n}\cdot\hat{j} = \frac{y}{2} \quad ; \quad dS_y = dS'\cdot\frac{y}{2} \Rightarrow dS = \frac{2\,dS_y}{y}$$

$$= \int_{x=-2}^{x=+2}\int_{z=0}^{z=2}\left(\frac{xz}{2}+\frac{xy}{2}\right)\cdot\frac{2\,dx\,dz}{y} = \int_{x=-2}^{x=+2}\int_{z=0}^{z=2}\left(\frac{zx}{y}+x\right)dx\,dz =$$

$$= \int_{x=-2}^{x=+2}\left[\frac{1\cdot x}{2y}\cdot z^2 + xz\right]_{z=0}^{z=2} dx = \int_{-2}^{+2}\left(\frac{2x}{y}+2x\right)dx =$$

$x^2+y^2=4$

$y^2=4-x^2$

$y=\sqrt{4-x^2}$

$$= \int_{-2}^{+2} \left(\frac{2x}{\sqrt{4-x^2}} + 2x \right) dx =$$

$$= \int_{-2}^{+2} \frac{2x}{\sqrt{4-x^2}}\, dx + \underset{\overset{\shortparallel}{0}}{\left. x^2 \right|_{-2}^{+2}} = - \int_{-2}^{+2} \frac{-2x}{\sqrt{4-x^2}}\, dx =$$

$$t^2 = 4 - x^2$$

$$2t\,dt = -2x\,dx$$

$$= - \int \frac{2\cancel{t}\,dt}{\cancel{t}} = -2t = -2\sqrt{4-x^2} \Big|_{-2}^{+2} = 0.$$

$$\Rightarrow \oint \vec{A}\cdot d\vec{S} = 24\pi$$

③ en cilíndricas

$$\hat{i} = \hat{r}\cos\theta - \operatorname{sen}\theta \cdot \hat{\theta}$$
$$\hat{j} = \hat{r}\operatorname{sen}\theta + \cos\theta \cdot \hat{\theta}$$

$x = r\cos\theta$

$y = r\operatorname{sen}\theta$

Ver. $\longrightarrow$

$$\vec{A}(r,\theta,z) = z\,(\cos\theta\,\hat{r} - \operatorname{sen}\theta\cdot\hat{\theta}) + r\cos\theta\,(\operatorname{sen}\theta\cdot\hat{r} + \cos\theta\,\hat{\theta}) + 3r^2\operatorname{sen}^2\theta\, z\cdot\hat{k}$$
$$= (z\cos\theta + r\cos\theta\operatorname{sen}\theta)\,\hat{r} + (r\cos^2\theta - \operatorname{sen}\theta)\,\hat{\theta} + 3r^2 z\operatorname{sen}^2\theta\,\hat{k}$$

$d\vec{S} = r\,d\theta\,dz\cdot\hat{r}$ en nuestro caso $r = 2$

$$\vec{A}\cdot d\vec{S} = (z\cos\theta + 2\cos\theta\operatorname{sen}\theta)\,2\,d\theta\,dz$$

$$\int \vec{A}\cdot d\vec{S} = \int_{z=0}^{z=2} dz \int_{\theta=0}^{\theta=2\pi} (2z\cos\theta + 4\cos\theta\operatorname{sen}\theta)\,d\theta = \int_{z=0}^{z=2} dz\,(0+0) = 0$$

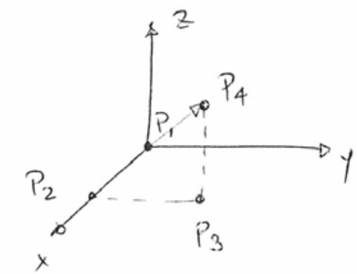
$$\int_S \vec{A}\cdot d\vec{S} = \int_V \vec{\nabla}\cdot\vec{A}\, dv = \int_v \overset{3y^2}{(\cancel{\quad})}\, dv = 3\int_{x=-2}^{x=+2}\int_{y=-\sqrt{4-x^2}}^{+\sqrt{4-x^2}}\int_{z=0}^{2} y^2\, dz\, dy\, dx$$

$$= 3\int_{x=-2}^{x=+2}\int_{y=-\sqrt{4-x^2}}^{y=+\sqrt{4-x^2}} 2y^2\, dy = 6\int_{x=2}^{y=+2}\left(\frac{1}{3}y^3\right)\Big|_{-\sqrt{}}^{+\sqrt{}} dx = 2\int_{x=-2}^{x=+2} 2\left(\sqrt{4-x^2}\right)^3 dx =$$

$$= 4\underbrace{\int_{x=-2}^{x=+2}\left(4-x^2\right)^{3/2} dx}_{6\pi} = 24\pi.$$

8.- Calcular la circulación del campo vectorial

$$\vec{A}(x,y,z) = (2x + y^2)\,\hat{\mathrm{x}} - 3yz\,\hat{\mathrm{y}} + \hat{\mathrm{z}}$$



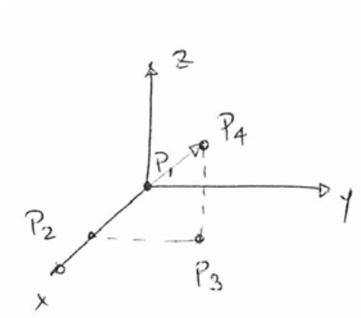
sobre la curva limitada por los puntos $P_1(0,0,0)$, $P_2(1,0,0)$, $P_3(1,1,0)$, y, $P_4(1,1,1)$.¿ Es conservativo el campo ?

P_1P_2 ; $\quad ① = \int \vec{A}\cdot\hat{i}\,dx = \int_{x=0}^{x=1} (2x+\underbrace{y^2}_{=0})\,dx = x^2\Big|_0^1 = 1$

$$② = \int_{(1,0,0)}^{(1,1,0)} \vec{A}\cdot\hat{j}\,dy = \int -3y\underbrace{z}_{=0 \ \forall \text{ trayectoria}}\,dy = 0$$

$$③ = \int \vec{A}\cdot\hat{k}\,dz = \int_{(1,1,0)}^{(1,1,1)} 1.\,dz = z\Big|_0^1 = 1$$

$$\boxed{\oint \vec{A}\cdot d\vec{\ell} = 2}$$



Hagamoslo de P_1 a P_4 directamente:

Ese camino responde al vector $(t, t, t) = (x, y, z)$

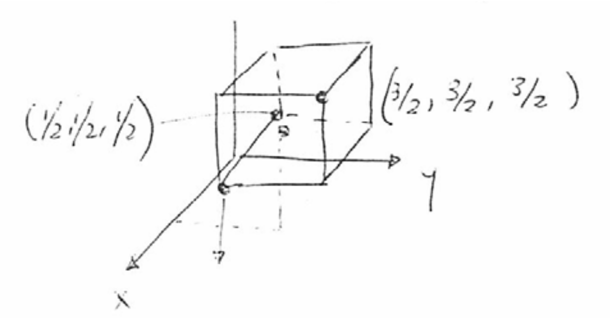
$$\int_0^1 \left((2t+t^2) - 3t^2 + 1 \right) dt = \int_0^1 \left(-2t^2 + 2t + 1 \right) dt =$$

$$= -\frac{2}{3}t^3 + \frac{1}{2}t^2 + t \Big|_0^1 = -\frac{2}{3} + \frac{1}{2} + 1 = \frac{-4+3+6}{6} = \boxed{\frac{5}{6}}$$

$\Rightarrow$ CAMPO NO CONSERVATIVO

9.- Dado el campo vectorial

$$\vec{A}(x,y,z) = x^2\,\hat{\mathrm{x}} + y^2\,\hat{\mathrm{y}} + z^2\,\hat{\mathrm{z}}$$



calcular el valor integral

$$\int_V \vec{\nabla}\cdot\vec{A}\; dV$$

siendo V el volumen de un cubo de arista unidad centrado en el punto de coordenadas (1,1,1) y de lados paralelos a los ejes.

$$\vec{\nabla}\cdot\vec{A} = 2x + 2y + 2z \;\longrightarrow\; \int_V 2(x+y+z)\,dv = 2\int_{x=1/2}^{x=3/2}\int_{y=1/2}^{y=3/2}\int_{z=1/2}^{z=3/2}(x+y+z)\,dx\,dy\,dz$$

$$\frac{3}{2}-\frac{1}{2}=1$$

$$=2\int_{1/2}^{3/2}dx\int_{1/2}^{3/2}dy\,\left(xz+yz+\frac{1}{2}z^2\right)\Big|_{z=1/2}^{z=3/2} = 2\iint\left[(x+y)+\frac{1}{2}\left(\frac{9}{4}-\frac{1}{4}\right)\right]dx\,dy =$$

$$= 2\int_{1/2}^{3/2} dx \int_{1/2}^{3/2} (x+y+1)\,dy = 2\int_{1/2}^{3/2} dx \underbrace{\left(xy + \frac{1}{2}y^2 + y \right)\Big|_{1/2}^{3/2}}_{\frac{3}{2}x + \frac{1}{2}\cdot\frac{9}{4} + \frac{3}{2} - \frac{1}{2}x - \frac{1}{2}\cdot\frac{1}{4} - \frac{1}{2} = x + 1 + 1} =$$

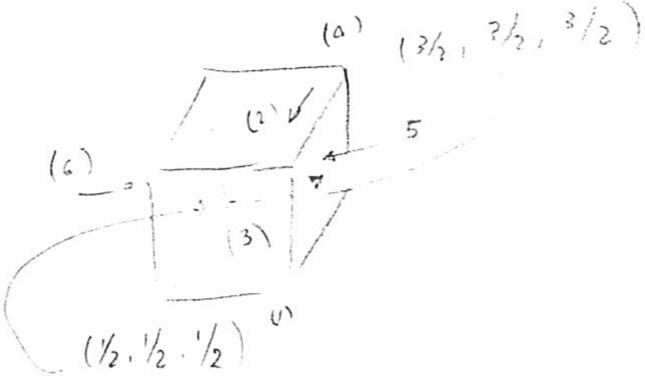
$$= 2\int_{1/2}^{3/2} (x+2)\,dx = 2\left[\left(\frac{1}{2}x^2 + 2x \right)\Big|_{1/2}^{3/2} \right] = 2\left[\frac{1}{2}\cdot\frac{9}{4} + \frac{6}{2} - \frac{1}{8} - \frac{2}{2} \right]$$

$$\cancel{= 2[4] = 8 \quad = 2[2+3] = 10} \quad = 2[1+2] = 6$$

Ver aplicando Tª. divergencia

$$\int_V \vec{\nabla}\cdot\vec{A}\, dV = \int_{S_c} \vec{A}\cdot d\vec{S} \qquad \vec{A} = x^2\hat{i} + y^2\hat{j} + z^2\hat{k}$$

Seis superficies



$$dS_1 = dS(-\hat{k})$$

$$\int \vec{A}\cdot d\vec{S}_1 = \int_{z=1/2} z^2\, dS = -\frac{1}{4}\underbrace{\int dS}_{=1} = -\frac{1}{4}$$

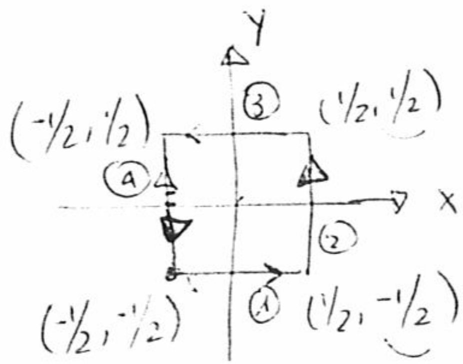
$$+ \quad \int \vec{A}\, d\vec{S}_2 = \int_{z=3/2} z^2\, dS = \frac{9}{4}\underbrace{\int dS}_{=1} = 9/4$$

$$\frac{9}{4} - \frac{1}{4} = 2$$

Análogo para las otras

$$\Rightarrow \int_{TOTAL} = 2\cdot 3 = 6$$

10.- Comprobar el teorema de Stokes para el campo vectorial $\vec{A}(x,y) = x\,\hat{y} + y\,\hat{z}$ sobre la superficie limitada por un cuadrado de lado unidad situado en el plano XOY, centrado en el origen, y de lados paralelos a los ejes de coordenadas.



T^h Stockes: $\oint_c \vec{A}\,\vec{dl} = \int_S (\vec{\nabla}\wedge\vec{A})\,\vec{dS}$

$$\oint \vec{A}\,\vec{dl} = \int_1 \vec{A}\cdot\vec{dl}_1 + \int_2 \vec{A}\cdot\vec{dl}_2 + \int_3 \vec{A}\cdot\vec{dl}_3 + \int_4 \vec{A}\,\vec{dl}$$

$\vec{dl}_1 = \hat{i}\,dx\,,\quad \vec{dl}_2 = \hat{j}\cdot dy$

$\vec{dl}_3 = -\hat{i}\,(dx)\,;\quad \vec{dl}_4 = -\hat{j}\cdot dy$

$$① = \int_{x=-1/2}^{x=1/2} 0 \, dx = 0 \quad ; \quad ② = \int_{y=-1/2}^{y=1/2} x \cdot dy = \frac{1}{2} \, y \Big|_{-1/2}^{1/2} = \frac{1}{2}\Big[\underbrace{\frac{1}{2} - \left(-\frac{1}{2}\right)}\Big] = \underline{1/2}$$

$y = -1/2 = cte$

$x = 1/2 = cte$

$$③ = ① = 0 \qquad ④ = \int_{y=1/2}^{y=-1/2} +x \, (+dy) = \frac{1}{2}(-y)\Big|_{+1/2}^{-1/2} = \frac{1}{2}\left(+\frac{1}{2} + \frac{1}{2}\right) = +1/2$$

$x = cte = -1/2$

$$\oint \vec{A} \, d\vec{\ell} = 1$$

$$\vec{\nabla} \wedge \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & x & y \end{vmatrix} = \hat{i}\,(1-0) + \hat{j}\,(0-0) + \hat{k}\,(1-0) = \underline{(\hat{i}+\hat{k})}$$

$$d\vec{S} = dx\,dy \cdot \hat{k}$$

$$\int_S 1 \cdot dx\,dy = \int_{-1/2}^{+1/2} dx \int_{-1/2}^{+1/2} dy = \int_{-1/2}^{+1/2} dx \cdot 1 = \frac{1}{2} - \left(-\frac{1}{2}\right) = \underline{\underline{1}}$$

11.- Verificar que el campo vectorial de componentes

$$\left(\frac{k}{x^2yz}, \frac{k}{xy^2z}, \frac{k}{xyz^2} \right) \quad \text{con} \quad k = \text{cte}$$

es conservativo. Determinar la función potencial.

Campo A cons. $\Rightarrow \nabla \wedge A = 0$

$$\int \vec{\nabla} \wedge \vec{A} = \int \vec{A} \cdot d\vec{l} = 0$$

$\vec{\nabla} \wedge \vec{A} = 0 \Rightarrow \vec{A}$ es conservativo $\Rightarrow$ buscamos una función potencial $\Rightarrow \nabla \wedge \nabla \phi = 0$.

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \hat{i}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \hat{j}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \hat{k}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

$$\frac{\partial A_z}{\partial y} = \frac{\partial}{\partial y}\left(\frac{k}{xz^2} \cdot \frac{1}{y}\right) = \frac{k}{xz^2}\left(\frac{-1}{y^2}\right)$$

$$\frac{\partial A_y}{\partial z} = \frac{\partial}{\partial z}\left(\frac{k}{xy^2} \cdot \frac{1}{z}\right) = \frac{k}{xy^2}\left(\frac{-1}{z^2}\right)$$

$$\hat{i}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) = \frac{-k}{xz^2y^2} - \frac{-k}{xy^2z^2} = 0$$

Análogo para las otras dos componentes.

◦ FUNCIÓN POTENCIAL

$$\vec{A} = -\vec{\nabla}\phi$$

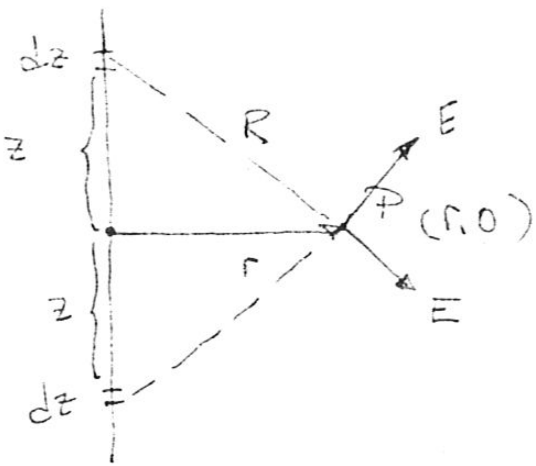
$$\left.\begin{array}{ll}
(1) & A_x = \dfrac{\partial\phi}{\partial x} \Rightarrow \phi = \displaystyle\int \frac{k}{x^2yz}\,dx = \frac{-k}{xyz} + cte(y,z) \\
(2) & A_y = \dfrac{\partial\phi}{\partial y} \Rightarrow \phi = \displaystyle\int \frac{k}{xy^2z}\,dy = \frac{-k}{xyz} + cte(x,z) \\
(3) & A_z = \dfrac{\partial\phi}{\partial z} \Rightarrow \phi = \displaystyle\int \frac{k}{xyz^2}\,dz = \frac{-k}{xyz} + cte(x,y)
\end{array}\right\}$$

Comparando las tres soluciones de ϕ, llegamos a la conclusión que $cte() = cte$ sin depender de ninguna de las tres variables

12.- Calcular el campo electrostático $\vec{E}$ originado por una distribución lineal e indefinida de carga $+\lambda_0$ = cte.

24.1 A la brava.

Coord. cilíndricas.



$$\vec{dE} = \frac{dQ = \lambda dz}{4\pi\varepsilon_0 R^2} \cdot \left(\frac{r\,\hat{a}_r - z\hat{a}}{\sqrt{r^2+z^2}} \right)$$

$R^2 = r^2 + z^2$

Notar que las comp. z se cancelan.

$$\boxed{\int \frac{dx}{(x^2+a^2)^{3/2}} = \frac{x}{a^2\sqrt{x^2+a^2}}}$$

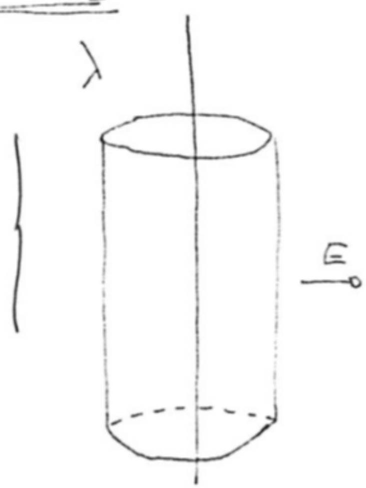
$$E = \int_{z=-\infty}^{+\infty} \frac{\lambda \cdot r\,dz}{4\pi\varepsilon_0 (r^2+z^2)^{3/2}}\,\hat{a}_r =$$

$$= \frac{\lambda \cdot r}{4\pi\varepsilon_0} \int_{-\infty}^{+\infty} \frac{dz}{(r^2+z^2)^{3/2}} \, \hat{a}_r = \frac{\lambda r}{4\pi\varepsilon_0}\left[\frac{z}{r^2 \cdot \sqrt{r^2+z^2}}\right]_{z=-\infty}^{z=+\infty} \hat{a}_r =$$

$$= \frac{\lambda r}{4\pi\varepsilon_0 r^2}\left[1-(-1)\right] = \frac{\lambda}{2\pi\varepsilon_0 r}$$

T. GAUSS

Prob. 18



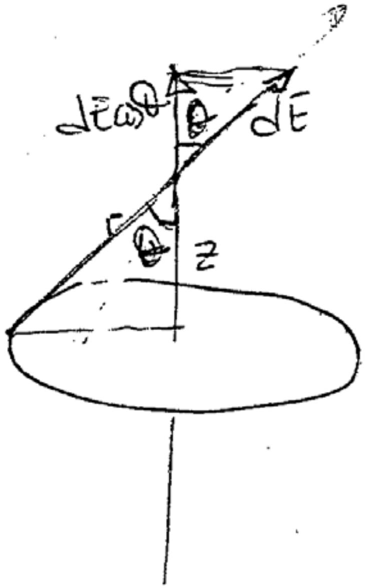
Notar que en las tapas $\vec{E} \perp \vec{n} \Rightarrow \vec{E} \cdot \vec{dS} = 0$

$$\int_{S_c} \vec{E} \cdot \vec{dS} = \frac{Q}{\varepsilon_0} \Rightarrow E \cdot 2\pi r L = \frac{\lambda \cdot L}{\varepsilon_0}$$

$$\Rightarrow E = \frac{1}{2\pi r} \frac{\lambda}{\varepsilon_0}$$

13.- Un anillo circular de radio R_0 está cargado con una densidad de carga uniforme $+\lambda_0$.

Calcular el campo y el potencial en un punto del eje de revolución.



Por simetría se anulan las componentes horizontales

$$dE_z = |d\vec{E}| \cos\theta$$

$$E_z = \int \frac{1}{4\pi\varepsilon_0} \frac{\lambda\, d\ell}{r^2} \cos\theta \qquad \wedge \ \cos\theta = \frac{z}{r}$$

$$\vec{E} = \hat{z} \frac{1}{4\pi\varepsilon_0} \int_c \frac{\lambda\, d\ell \cdot z}{r^3} = \hat{z} \frac{\lambda\, z}{4\pi\varepsilon_0} \frac{1}{r^3} \underbrace{\int_c d\ell}_{2\pi R_0} =$$

$$= \frac{\lambda z}{4\cancel{\pi}\varepsilon_0} \frac{2\cancel{\pi} R_0}{r^3} \hat{z} \qquad \wedge \quad r^2 = R_0^2 + z^2$$

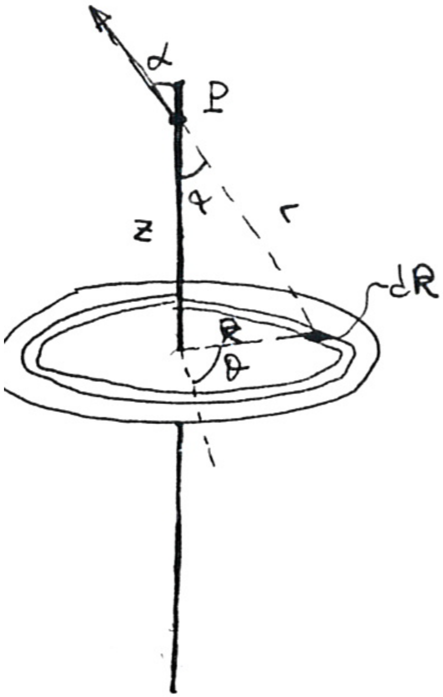
$$\boxed{\vec{E} = \frac{\lambda z R_0}{2\varepsilon_0 (R_0^2 + z^2)^{3/2}} \hat{z}}$$

Potencial $\quad \phi = -\int \vec{E} \cdot \vec{dr} \quad ; \quad \vec{dr} = \hat{x} dx + \hat{y} dy + \hat{z} dz$

$$\phi = \frac{-\lambda R_0}{2\varepsilon_0} \int \frac{z\, dz}{(R_0^2 + z^2)^{3/2}} = \frac{\lambda R_0}{2\varepsilon_0} \frac{1}{[z^2 + R_0^2]^{1/2}} + c^{te}$$

Si $z = \infty \Rightarrow \phi = 0 \Rightarrow$ $\boxed{c^{te} = 0}$

14.- Una superficie circular de radio R_0 está cargada con una distribución uniforme $+\sigma_0$. Calcular el campo $\overrightarrow{E}$ en un punto del eje de revolución.



$$dE = \frac{1}{4\pi\varepsilon_0}\,\frac{dq}{r^2}$$

$$dq = \sigma . dS = \sigma_0 . R\,dR\,d\varphi$$

Por simetría, las componentes horizontales se anulan. Las verticales $dE_\perp = dE \cos\alpha$

$$E_z = \frac{1}{4\pi\varepsilon_0}\int \frac{dq}{r^2} . \cos\alpha =$$

$$= \frac{1}{4\pi\varepsilon_0}\int \frac{\sigma_0 . R\,dR\,d\varphi}{R^2+z^2} . \frac{z}{\sqrt{R^2+z^2}} =$$

$$= \frac{\sigma_0 z}{4\pi\varepsilon_0}\int_0^{2\pi} d\varphi \int_0^{R_0} \frac{R\,dR}{(R^2+z^2)^{3/2}} = \frac{2\cancel{\pi}\sigma_0 z}{4\cancel{\pi}\varepsilon_0}\left[\frac{-1}{\sqrt{R^2+z^2}}\right]_0^{R_0} =$$

<u>NOTA</u>: $\displaystyle\int \frac{x\,dx}{(x^2+a^2)^{3/2}} = \frac{-1}{\sqrt{x^2+a^2}}$

$$= \frac{\sigma_0 \cdot z}{2\varepsilon_0}\left[\frac{1}{z} - \frac{1}{\sqrt{R_0^2+z^2}}\right]$$

$$\vec{E} = \frac{\sigma_0}{2\varepsilon_0}\left[1 - \cos\alpha\right]\hat{z}$$

15.- Resolver el problema electrostático semejante al anterior, para una distribución de carga uniforme en una corona circular de radios R_1 y R_2.

Particularizar el resultado obtenido a la situación donde $R_1 \rightarrow 0$, y, a la vez $R_2 \rightarrow \infty$.

- *Es similar al anterior, salvo que los límites de la integral ahora será desde R_1 hasta R_2.*

$$\vec{E} = \frac{\sigma_0 \cdot z}{2\varepsilon_0}\left[\frac{1}{\sqrt{R_1^2+z^2}} - \frac{1}{\sqrt{R_2^2+z^2}}\right]\hat{z}$$

Si $R_1 \rightarrow 0 \Rightarrow$ $$\vec{E} = \frac{\sigma_0 z}{2\varepsilon_0}\left[\frac{1}{z} - \frac{1}{\sqrt{R_2^2+z^2}}\right]\hat{z}$$

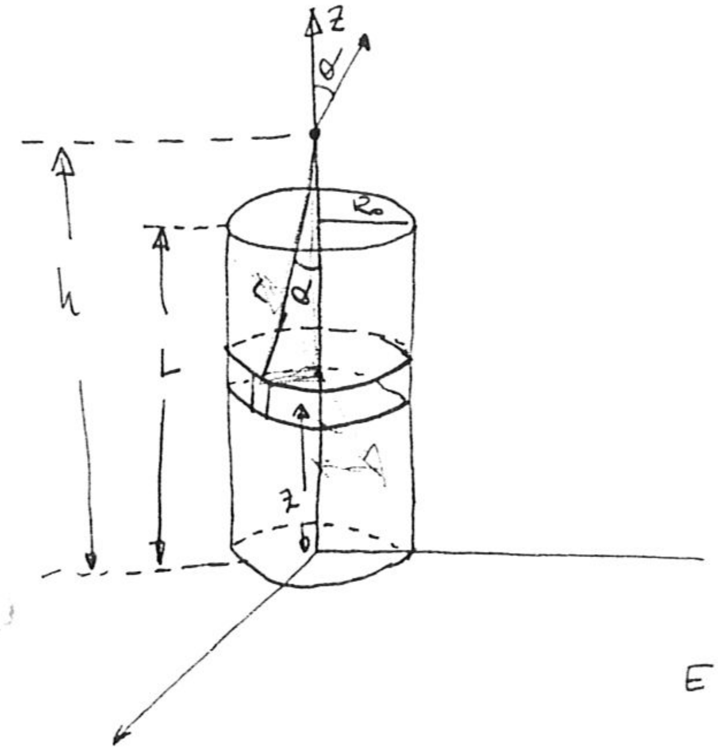
$R_2 \rightarrow \infty \Rightarrow$ $$\vec{E} = \frac{\sigma_0 z}{2\varepsilon_0}\,\frac{1}{\sqrt{R_1^2+z^2}}\,\hat{z}$$

$$R_1 \to 0\ ,\ R_2 \to \infty \Rightarrow \quad \vec{E} = \frac{\sigma_0 \cdot \cancel{z}}{2\varepsilon_0 \cancel{z}}\,\hat{z} \quad = \frac{\sigma_0}{2\varepsilon_0}\,\hat{z}$$

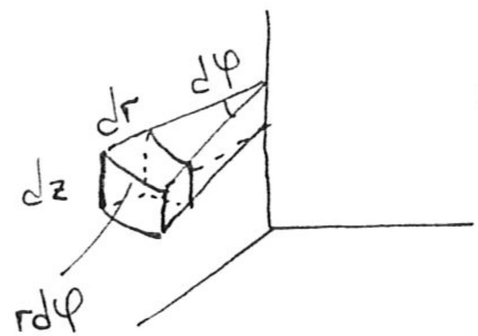
Igual que para un plano. Plano infinito = disco infinito

16.- Una superficie cilíndrica de radio R_0 y altura L, tiene una carga distribuida uniformemente.

Calcular el campo $\overrightarrow{E}$ en cualquier punto del eje de revolución.



Tomamos coord. cilíndricas



$$dV = r\,dr\,d\varphi\,dz$$

$$dS = R_0\,d\varphi\,dz.$$

$$E = \int dE \cos\theta = \int \frac{\sigma\,dS}{4\pi\varepsilon_0}\,\frac{1}{r^2}\cos\theta$$

como $\cos\theta = \dfrac{h-z}{r} \quad \wedge \quad r = \left[R_0^2 + (h-z)^2\right]^{1/2} \to r$

$$E = \int_0^L dz \int_0^{2\pi} \frac{\sigma R_0}{4\pi\varepsilon_0} \frac{h-z}{\left[R_0^2 + (h-z)^2\right]^{1/2}} \, d\varphi =$$

$$\overset{*}{=} \frac{2\cancel{\pi}\sigma R_0}{2\cancel{4\pi}\varepsilon_0} \int_0^L \frac{h-z}{\left[R_0^2 + (h-z)^2\right]^{3/2}} dz \overset{\wedge}{=}$$

$\wedge$ Sea $x = h - z \Rightarrow dx = -dz$

$$\int_0^L \frac{h-z}{\left[R_0^2 + (h-z)^2\right]^{3/2}} dz = \int_h^{h-L} \frac{-x \; dx}{\left[R_0^2 + x^2\right]^{3/2}} = \int_{h-L}^{h} \frac{x \, dx}{\left[R_0^2 + x^2\right]^{3/2}} =$$

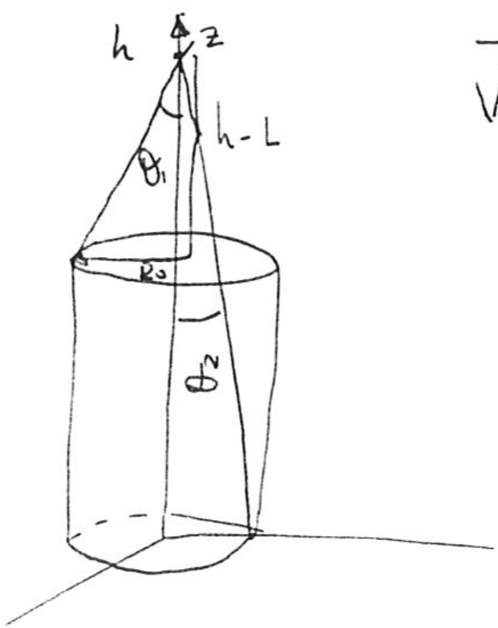
$$= \frac{-1}{\sqrt{x^2 + R_0^2}} \Bigg|_{h-L}^{h} = \left[\frac{1}{\sqrt{(h-L)^2 + R_0^2}} - \frac{1}{\sqrt{h^2 + R_0^2}} \right]$$

$$= \frac{\sigma R_0}{2\varepsilon_0}\left[\frac{1}{\sqrt{(h-L)^2+R_0^2}} - \frac{1}{\sqrt{h^2+R_0^2}}\right] = \boxed{\frac{\sigma}{2\varepsilon_0}\left(\operatorname{sen}\theta_1 - \operatorname{sen}\theta_2\right) = E_z}$$

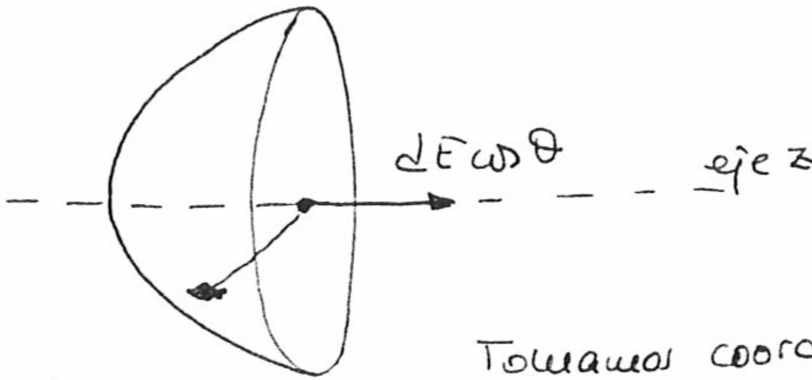
(si $h>L$)

$$\wedge \quad \frac{R_0}{\sqrt{(h-L)^2+R_0^2}} = \operatorname{sen}\theta_1$$

$$\frac{R_0}{\sqrt{h^2+R_0^2}} = \operatorname{sen}\theta_2$$

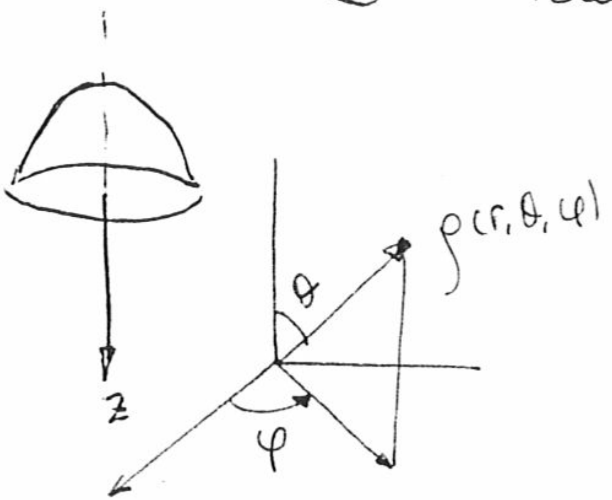


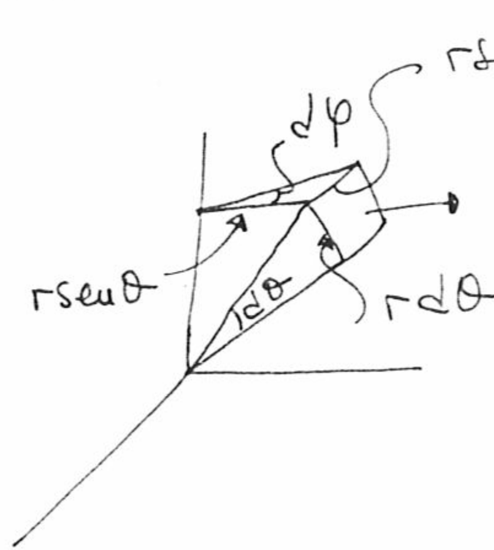
17.- Una superficie en forma de hemiesfera posee una distribución de carga positiva σ_0 = cte.
Calcular el campo electrostático en el centro de la hemiesfera.



$$E = \int dE \cos\theta$$

Tomamos coord esféricas





$$dS = r^2 \operatorname{sen}\theta \, d\varphi \, d\phi$$

$$r = R_0.$$

$$E = \int \frac{\sigma\, dS}{4\pi\varepsilon_0}\,\frac{1}{R_0^2}\cos\theta = \int\!\!\int \frac{\sigma\, d\cancel{S}}{4\pi\varepsilon_0}\,\frac{1}{\cancel{R_0^2}}\cos\theta\,\cancel{R_0^2}\,\mathrm{sen}\theta\, d\theta\, d\varphi$$

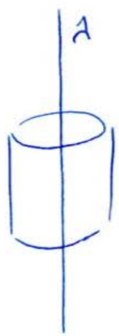
$$= \int_0^{\pi/2} d\theta \cdot \cancel{2\pi}\, \frac{\cancel{2}\cancel{\pi}\sigma}{\cancel{4}\cancel{\pi}\varepsilon_0}\,\mathrm{sen}\theta\cos\theta\, d\theta =$$

$x = \mathrm{sen}\,\theta$

$dx = \cos\theta\, d\theta$

$$= \frac{\sigma}{2\varepsilon_0}\int_0^1 x\, dx = \frac{\sigma}{2\varepsilon_0}\,\frac{x^2}{2}\Big|_0^1 = \boxed{\frac{\sigma}{4\varepsilon_0} = E_z}$$

18.- Calcular el campo electrostático originado por dos distribuciones de carga lineales, paralelas e indefinidas, con densidades $+\lambda_0$ y $-\lambda_0$ situadas a una distancia *d* en el vacío.



Tomamos como superficie para aplicar el T.ª Gauss un cilindro coaxial con el hilo. Por simetría $\vec{E} = E(r)\cdot\hat{r}$

$$\int_{S_c} \vec{E}\cdot d\vec{s} = \int_{S.\,lateral} \vec{E}\cdot d\vec{s} = E(r)\cdot 2\pi r \cancel{L} = \frac{\lambda \cancel{L}}{\varepsilon_0}$$

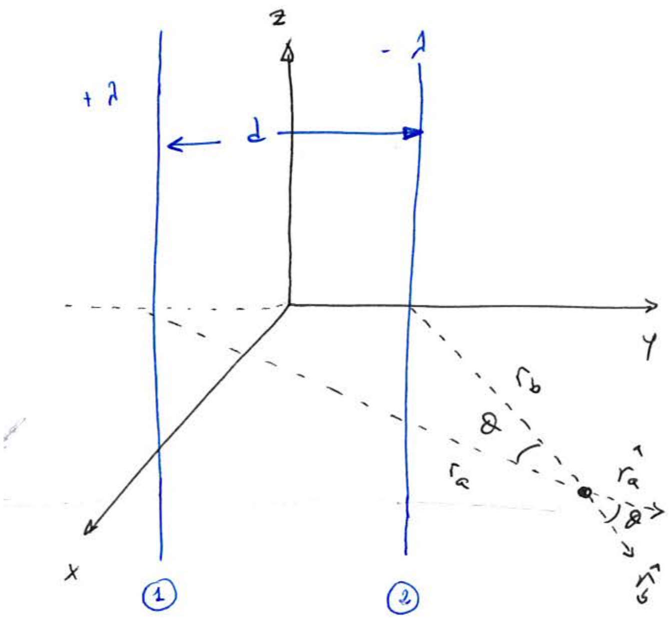
$$\vec{E} = \frac{\lambda}{2\pi\varepsilon_0 r}\cdot\hat{r}$$

$$\phi(r) = -\int E(r)\cdot dr =$$

$$\phi(r) = -\frac{\lambda}{2\pi\varepsilon_0}\ln r + C$$

$$\phi(r) = \frac{-\lambda}{2\pi\cdot\varepsilon_0}\ln(C'\cdot r)$$

Sea ahora el sistema formado por dos distribuciones de carga



$$\vec{E}_1 = \frac{\lambda}{2\pi\varepsilon_0}\frac{\hat{r}_a}{r_a}$$

$$\vec{E}_2 = \frac{-\lambda}{2\pi\varepsilon_0}\frac{\hat{r}_b}{r_b}$$

$$\vec{E} = \vec{E}_1 + \vec{E}_2 = \frac{\lambda}{2\pi\varepsilon_0}\left\{\frac{\hat{r}_a}{r_a} - \frac{\hat{r}_b}{r_b}\right\}$$

$$|\vec{E}|^2 = \vec{E}\cdot\vec{E} = \left(\frac{\lambda}{2\pi\varepsilon_0}\right)^2\left\{\frac{1}{r_a^2} + \frac{1}{r_b^2} - \frac{2\hat{r}_a\cdot\hat{r}_b}{r_a r_b}\right\} =$$

$$\hat{r}_a\cdot\hat{r}_b = \cos\theta$$

$$|E|^2 = \left(\frac{\lambda}{2\pi\varepsilon_0}\right)^2\left\{\frac{r_a^2 + r_b^2 - 2r_a r_b\cos\theta}{r_a^2\cdot r_b^2}\right\} = \left(\frac{\lambda\cdot d}{2\pi\varepsilon_0 r_a r_b}\right)^2$$

Th. Coseno

$$\Rightarrow \boxed{|\vec{E}| = \frac{\lambda\cdot d}{2\pi\varepsilon_0 r_a\cdot r_b}}$$

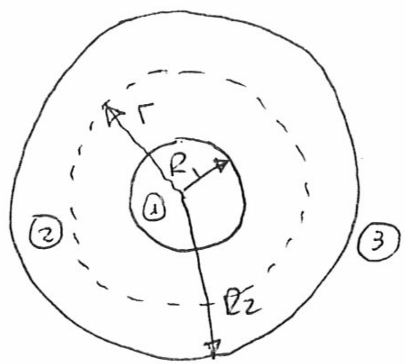
Si se quiere expresar de forma exacta $\vec{E} = \vec{E}_1 + \vec{E}_2$

$$\vec{E}_1 = \frac{\lambda}{2\pi\varepsilon_0} \frac{x \cdot \hat{x} + \left(y + \frac{d}{2}\right)\hat{y}}{x^2 + \left(y + \frac{d}{2}\right)^2}$$

$$\vec{E}_2 = \frac{-\lambda}{2\pi\varepsilon_0} \frac{x \cdot \hat{x} + \left(y - d/2\right)\hat{y}}{x^2 + \left(y - \frac{d}{2}\right)^2}$$

19.- El espacio comprendido entre dos superficies esféricas de radios R_1 y R_2, contiene una carga definida por la función $\rho(r) = A/r^2$, siendo A una constante positiva.

Suponiendo que la superficie interior ($r = R_1$) se encuentra a potencial V_0, determinar el campo $\vec{E}$ y potencial en todo el espacio.



$\rho(r) = \dfrac{A}{r_2}$ $(R_1 < r < R_2)$

$\rho(r) = 0$ resto.-

ZONA 1

$$\oint \vec{E} \cdot \vec{dS} = \frac{q}{\varepsilon_0} = 0.-$$

$$\parallel \quad \vec{E} \parallel \vec{dS}$$

$$\vec{E} \underbrace{\oint \vec{dS}}_{\neq 0} = 0 \Rightarrow \boxed{E = 0}$$

ZONA 2

$$\int \vec{E} \cdot \vec{ds} = \frac{q}{\varepsilon_0} = E \int_S ds = E \cdot \overbrace{4\pi r^2}^{\text{Sup. esfera}} = \frac{q}{\varepsilon_0}$$

Puedo poner ε_0

$$dV = r^2 \operatorname{sen}\theta \, d\theta \, d\varphi \, dr$$

Hallemos q

$$q = \int_V \rho(r) \, dV = \iiint dr \, d\theta \, d\varphi \, \frac{A}{\cancel{r^2}} \cancel{r^2} \operatorname{sen}\theta =$$

$$= \int_{r=R_1}^{r} dr \int_0^{\pi} \operatorname{sen}\theta \, d\theta \cdot A \, 2\pi = A \, 2\pi \int_{R_1}^{r} dr \left[-\cos\theta\right]_0^{\pi} =$$

$$= A \, 2\pi \left[r - R_1\right] \left[-(-1) + 1\right] = 4\pi A (r - R_1)$$

$$E \cancel{4\pi} r^2 = \frac{\cancel{4\pi} A (r - R_1)}{\varepsilon_0} \Rightarrow$$

$$\boxed{E = \frac{A (r - R_1)}{r^2 \varepsilon_0}}$$

3 Análogo a zona 2 salvo que r ahora varía desde R_1 a R_2

$$\Rightarrow \quad q = 4\pi A (R_2 - R_1) \Rightarrow$$

$$E = \frac{A(R_2 - R_1)}{r^2 \varepsilon_0} = \frac{q}{4\pi\varepsilon_0} \frac{1}{r^2}$$

Como si fuese una carga puntual del mismo valor *q* que la carga total y estuviese situada en el centro de la distribución.

b) POTENCIAL

$$d\vec{r} = dr.\hat{r} + r d\theta.\hat{\theta} + r \operatorname{sen}\theta d\varphi \hat{\varphi}$$

Como $\phi = -\int \vec{E} d\vec{r}$ y $\vec{E}$ solo tiene componente radial $\Rightarrow$

$$\phi = -\int E dr$$

ZONA 1 $\Rightarrow$ $\phi = cte.$ –

ZONA 2

$$\phi_2 = -\int \frac{A(r - R_1)}{\varepsilon_0 r^2} dr = -\frac{A}{\varepsilon_0} \left\{ \ln r + \frac{R_1}{r} \right\} + cte$$

ONA 3

$$\phi_3 = -\frac{A(R_2-R_1)}{\varepsilon_0}\int \frac{dr}{r^2} = \frac{A}{\varepsilon_0 r}(R_2-R_1) + cte'$$

Si $r \rightarrow \infty$ $\quad \phi_3 \longrightarrow 0 \Rightarrow cte'' = 0.$ —

Las demas las determinamos por continuidad:

$$\phi_3(R_2) = \phi_2(R_2) \quad , \quad \phi_2(R_1) = \phi_1(R_1)$$

$$\frac{A}{\varepsilon_0 R_2}(R_2-R_1) = -\frac{A}{\varepsilon_0}\left\{ \ln R_2 + \frac{R_1}{R_2} \right\} + cte' \Rightarrow$$

$$= \frac{A}{\varepsilon_0}\left\{ \ln R_2 + \frac{R_1}{R_2} + \frac{1}{\varepsilon_r}\left(1 - \frac{R_1}{R_2}\right)\right\} \qquad \text{if } \varepsilon_r = 1 \Rightarrow \varepsilon =$$

$$cte' = \frac{A}{\varepsilon_0}\left(1 + \ln R_2\right)$$

CASO PARTICULAR $\varepsilon_r = 1$

$$-\frac{A}{\varepsilon_0}\left\{\ln R_1 + 1\right\} + \frac{A}{\varepsilon_0}\left(1 + \ln R_2\right) = \text{CTE.} -$$

$$\Rightarrow \text{CTE} = -\frac{A}{\varepsilon_0}\left\{\ln R_1 - \ln R_2\right\} = -\frac{A}{\varepsilon_0}\ln\frac{R_1}{R_2} =$$

$$\boxed{= \frac{A}{\varepsilon_0}\ln\frac{R_2}{R_1} = \phi_1}$$

$$-\frac{A}{\varepsilon}\left\{\ln R_1 + 1\right\} + \frac{A}{\varepsilon_0}\left\{\ln R_2 + \frac{R_1}{R_2} + \frac{1}{\varepsilon_r}\left(1 - \frac{R_1}{R_2}\right)\right\} = \text{CTE}$$

$$\frac{A}{\varepsilon_0}\left\{-\frac{\ln R_1}{\varepsilon_r} - \frac{1}{\varepsilon_r} + \ln R_2 + \frac{R_1}{R_2} + \frac{1}{\varepsilon_r}\left(1 - \frac{R_1}{R_2}\right)\right\} =$$

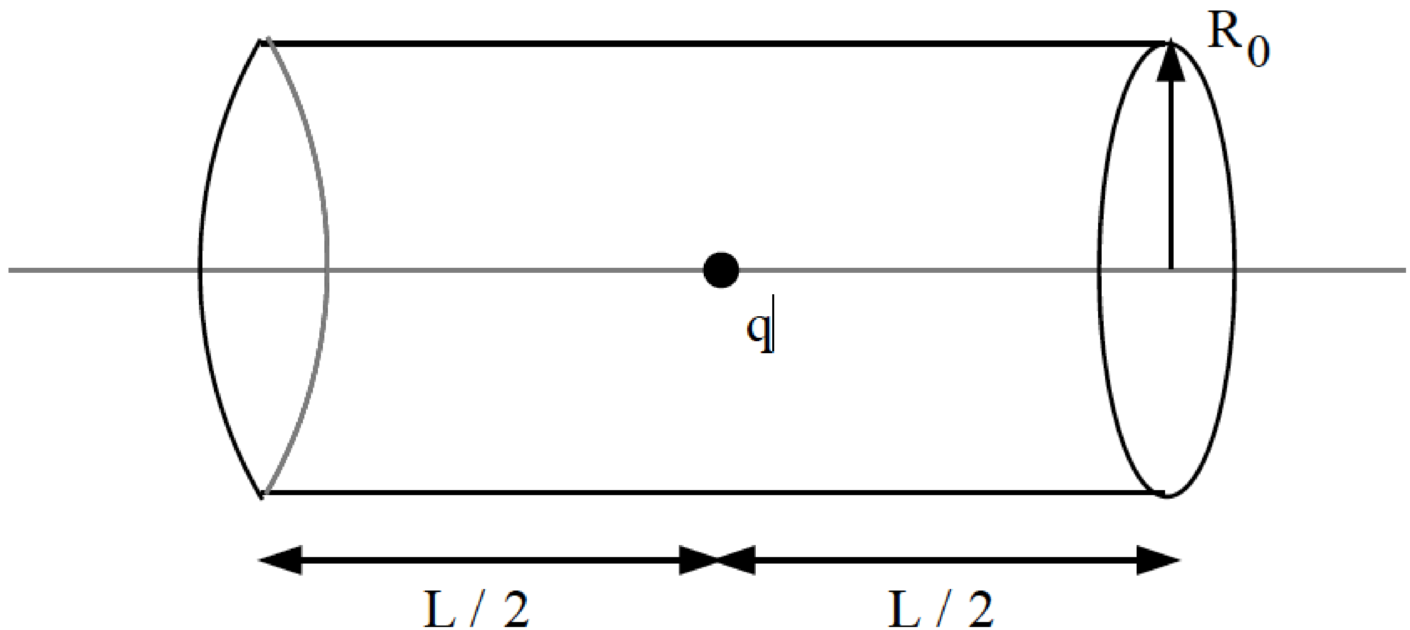
$$= \frac{A}{\varepsilon_0}\left\{\ln R_2 + \frac{R_1}{R_2} - \frac{1}{\varepsilon_r}\left(\ln R_1 - \frac{R_1}{R_2}\right)\right\} = \text{CTE} = \phi_1$$

if $\varepsilon_r = 1$

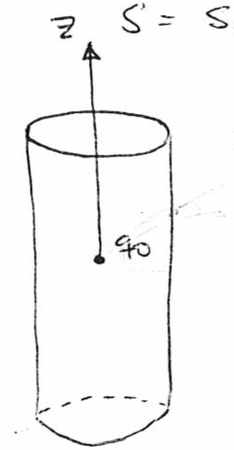
20.- Calcular el campo originado por una distribución de carga uniforme e indefinida de densidad ρ_0 comprendida entre dos cilindros coaxiales de radios $R_1 = R_0$, y $R_1 = 2\,R_0$.

Considerar la superficie de radio R_1 a un potencial V_0 constante.

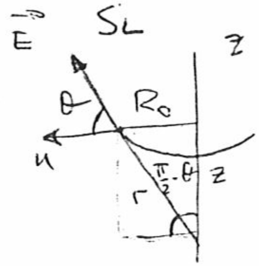
21.- Comprobar el teorema de Gauss para una carga puntual q, localizada en el interior de la superficie de forma cilíndrica como muestra la figura.



Tenemos que calcular el flujo a través de las tres superficies (dos tapas + superficie lateral)



$S = S_1 + S_2 + S_L$



$$\phi_L = \int_{S_L} \vec{E}\cdot d\vec{S} = \int E\, dS \cos\theta =$$

$$= \int_{S_L} \frac{q_0}{4\pi\varepsilon_0 r^2}\cdot\frac{R_0}{r}\, dS =$$

$$\cos\theta = \text{sen}\left(\frac{\pi}{2}-\theta\right) = \frac{R_0}{r}$$

$$r = \sqrt{z^2 + R_0^2}$$

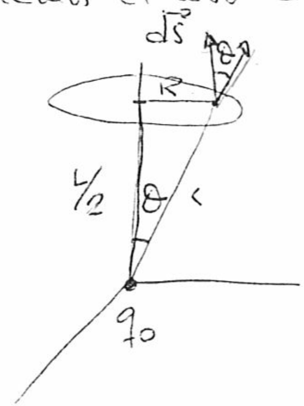
$$\phi_L = \int_{S_L} \frac{q_0 R_0}{4\pi\varepsilon_0}\,\frac{1}{\left[\sqrt{z^2+R_0^2}\right]^3}\,\overbrace{dz\,R_0\,d\varphi}^{dS = R_0\,dz\,d\varphi} = \frac{q_0 R_0^2}{4\pi\varepsilon_0}\int_0^{2\pi} d\varphi \int_{-L/2}^{+L/2} \frac{dz}{\left(z^2+R_0^2\right)^{3/2}} =$$

$$= \frac{q_0 \cancel{R_0^2}}{\cancel{4}_{2}\cancel{\pi}\,\varepsilon_0}\,\cancel{2\pi}\,\frac{L}{\cancel{R_0^2}\left(\frac{L^2}{4}+R_0^2\right)^{1/2}} =$$

$$\boxed{\text{NOTA: } \int \frac{dx}{\left(x^2+a^2\right)^{3/2}} = \frac{x}{a^2\sqrt{a^2+x^2}}}$$

$$\phi_L = \frac{q_0}{2\varepsilon_0}\,\frac{L}{\left(\frac{L^2}{4}+R_0^2\right)^{1/2}}$$

Veamos el caso de las bases.



$$\phi_1 = \int E\, dS \cos\theta = \int_{S_1} \frac{q_0}{4\pi\varepsilon_0 r^2}\cdot\frac{L/2}{r}\, dS =$$

$$= \int_0^{2\pi} d\varphi \int_{R=0}^{R_0} \frac{q_0\, L/2}{4\pi\varepsilon_0}\cdot\frac{R}{\left[\frac{L^2}{4}+R^2\right]^{3/2}}\, dR =$$

NOTA $\left[\int \frac{x\,dx}{[a^2+x^2]^{3/2}} = \frac{-1}{\sqrt{x^2+a^2}}\right]$

$$= \cancel{2\pi}\,\frac{q_0 \cdot L}{\cancel{2}\cdot 4\pi\varepsilon_0}\,\frac{-1}{\left[\frac{L^2}{4}+R^2\right]^{1/2}}\Bigg|_{R=0}^{R=R_0} = \frac{q_0 L}{4\varepsilon_0}\left[\frac{-1}{\left(\frac{L^2}{4}+R_0^2\right)^{1/2}}+\frac{1}{\sqrt{L^2/4}}\right]$$

Para la otra base, por simetría, tenemos el mismo resultado ⇒

$$\phi_T = 2\phi_1+\phi_L = \frac{q_0L}{2\varepsilon_0}\left[\frac{-1}{\left(\frac{L^2}{4}+R_0^2\right)^{1/2}}+\frac{2}{L}\right]+\frac{q_0L}{2\varepsilon_0}\left[\frac{1}{\left(\frac{L^2}{4}+R_0^2\right)^{1/2}}\right] = \frac{q_0}{\varepsilon_0}$$ #